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1982 J. Phys. A: Math. Gen. 15 3001

(http://iopscience.iop.org/0305-4470/15/9/046)

J. Phys. A: Math. Gen. **15** (1982) 3001–3005. Printed in Great Britain

COMMENT

Superconductivity for relativistic electrons

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Received 18 December 1981, in final form 8 February 1982

Abstract. Ginzburg–Landau equations for the pairing of relativistic electrons are derived. It is observed that relativistic effects tend to make superconductors type I, and tend to make the superconducting phase transition strongly first order.

In recent papers (Barrois 1979, Bailin and Love 1981a, b, 1982a), the Cooper pairing of relativistic quarks has been studied using the Dyson equation for the proper self-energy of the quark. (This technique was originally developed by Nambu (1960) for the case of non-relativistic electrons.) In the present note, we apply this technique to derive Ginzburg–Landau equations for the pairing of relativistic electrons in a BCS approximation, and discuss whether the superconductor is likely to be type I or type II. We also discuss the superconducting phase transition for relativistic electrons, taking account of fluctuations in the electromagnetic field. A similar discussion has been given for the case of quark matter (Bailin and Love 1981c). However in that case it is fluctuations in the colour gluon fields that are important. Also the $J^P = 0^+$ quark Cooper pairs have non-trivial internal symmetry properties coming from colour and flavour. Consequently, the results for electrons differ by various group theory factors from the results for quarks. In addition, the Cooper pairing of quarks is due to colour gluon exchange where as the pairing we have in mind here is due to phonon exchange. This leads to differences in the connection between the detailed gap matrix and the order parameter. Our results may be of significance for astrophysical situations, or, if extremely precise measurements can be made, for superconductivity in heavy metals in the laboratory, although in this case the relativistic corrections will be masked by strong coupling corrections to BCS theory. They can also be useful in studying the implications for non-relativistic superconductors of relativistic theories such as the Glashow–Weinberg–Salam model of electroweak interactions (Bailin and Love 1982b).

The gap matrix Δ may be introduced as an effective Lagrangian term

$$\mathscr{L}_\Delta = \int \mathrm{d}^4x\, \mathrm{d}^4y [\bar{e}_c(x)\Delta(x, y)e(y) + \mathrm{HC}], \tag{1}$$

where e_c is the charge conjugate electron field. The inverse electron propagator for the Cooper paired system is then

$$S^{-1}(p', p) = \begin{pmatrix} (\not{p} - m + \mu\gamma_0)(2\pi)^4\delta(p-p') & \gamma^0\Delta^\dagger(p, p')\gamma^0 \\ \Delta(p', p) & (\not{p} - m - \mu\gamma_0)(2\pi)^4\delta(p-p') \end{pmatrix} \tag{2}$$

0305-4470/82/093001+05$02.00

where $\Delta(p', p)$ is the momentum-space gap matrix, and

$$\mu = (p_F^2 + m^2)^{1/2} \tag{3}$$

is the chemical potential. The very small correction to μ from the work function in the case of a metal has been neglected. $S^{-1}(p', p)$ is a 2×2 matrix which acts on $\begin{pmatrix} e \\ e_c \end{pmatrix}$, and Δ is a matrix in spinor space. For the diagonal entries of S^{-1}, p and p' are the momenta of incoming and outgoing electrons, and for the off-diagonal entries both electrons are incoming (or outgoing) with momenta p and $-p'$. It will sometimes be convenient to consider Δ as a function of the variables

$$k = \tfrac{1}{2}(p + p'), \qquad K = p' - p. \tag{4}$$

If we write

$$S(p', p) = \begin{pmatrix} A(p', p) & B(p', p) \\ C(p', p) & D(p', p) \end{pmatrix}, \tag{5}$$

then the off-diagonal element $C(p', p)$ is needed to derive the gap equation. This has been evaluated in Bailin and Love (1982a). The gap equation may be obtained from the (off-diagonal component of) the Dyson equation shown in figure 1. The Matsubara frequency sum may be converted to an integral round an anticlockwise contour which includes the poles of $C(p'-p+q, q)$ but *not* those of $\tanh(\frac{1}{2}\beta q_0)$, where $\beta = (k_B T)^{-1}$. Thus

$$\Delta(p', p) = \tfrac{1}{2} i g^2 \int \mathrm{d}^4 q (2\pi)^{-4} D_{AB}(p-q) \tilde{\Gamma}^A C(p'-p+q, q) \Gamma^B \tanh(\tfrac{1}{2}\beta q_0). \tag{6}$$

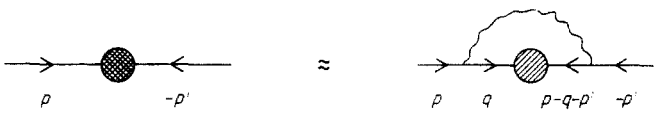

Figure 1. Single phonon exchange contribution to the Dyson equation. Cross hatching denotes the proper self-energy, and diagonal marking the exact propagator of the electron.

For generality, we have taken the interaction vertices to be $-ig\Gamma^A$ where Γ^A is some Dirac covariant, and the propagator of the exchanged phonon to be $D_{AB}(p-q)$ where A and B are spin indices.

$$\tilde{\Gamma}^A \equiv \mathscr{C}(\Gamma^A)^{\mathrm{T}}\mathscr{C}^{-1}, \tag{7}$$

where $\mathscr{C}$ is the charge conjugation matrix with the property

$$\mathscr{C}\gamma_\mu^{\mathrm{T}}\mathscr{C}^{-1} = -\gamma_\mu. \tag{8}$$

Keeping only leading terms in $[(\boldsymbol{q}^2 + m^2)^{1/2} - \mu]/\mu$, assuming that $|\boldsymbol{K}| \ll |\boldsymbol{q}|$ (distance scales large compared with p_F^{-1}), and expanding in powers of $\boldsymbol{K}$ up to order $\boldsymbol{K}^2$, evaluation of the residues gives

$$\Delta(\boldsymbol{k}, \boldsymbol{K}) = \tfrac{1}{2} g^2 \int \mathrm{d}^3 q (2\pi)^{-3} D_{AB}(\boldsymbol{k}-\boldsymbol{q}) \tilde{\Gamma}^A R \Gamma^B, \tag{9}$$

where

$$D_{AB}(\boldsymbol{k}-\boldsymbol{q}) \equiv D_{AB}(k_0-q_0=0, \boldsymbol{k}-\boldsymbol{q}), \tag{10}$$

$$\Delta(\boldsymbol{k}, \boldsymbol{K}) \equiv \Delta(k_0=0, \boldsymbol{k}, K_0=0, \boldsymbol{K}) \tag{11}$$

and

$$R = \bar{\Delta}(\boldsymbol{q}, \boldsymbol{K})[E_-^2 I + \tilde{\Delta}(\boldsymbol{q}, \boldsymbol{K})\bar{\Delta}(\boldsymbol{q}, \boldsymbol{K})]^{-1/2} \tanh[\tfrac{1}{2}\beta(E_-^2 I + \tilde{\Delta}\bar{\Delta})^{1/2}]$$
$$-\tfrac{1}{16}(\boldsymbol{q}\cdot\boldsymbol{K})^2(\beta^2/\mu^2 E_-)\tanh(\tfrac{1}{2}\beta E_-)\operatorname{sech}^2(\tfrac{1}{2}\beta E_-)\bar{\Delta}(\boldsymbol{q}, \boldsymbol{K}) \tag{12}$$

with the notation

$$q \equiv (\mu, \boldsymbol{q}), \qquad \tilde{q} \equiv (\mu, -\boldsymbol{q}), \qquad E_- \equiv (\boldsymbol{q}^2+m^2)^{1/2}-\mu \tag{13}$$

and

$$\bar{\Delta}(\boldsymbol{q}, \boldsymbol{K}) = (1/4\mu^2)(\tilde{q}-m)\Delta(\boldsymbol{q}, \boldsymbol{K})(q+m)$$
$$\tilde{\Delta}(\boldsymbol{q}, \boldsymbol{K}) = \gamma^0\Delta^+(\boldsymbol{q}, \boldsymbol{K})\gamma^0. \tag{14}$$

For $J^P = 0^+$ pairing the most general electron gap matrix consistent with Fermi statistics is

$$\Delta(\boldsymbol{n}, \boldsymbol{K}) = \Delta_1\gamma_5 + \Delta_2\boldsymbol{n}\cdot\boldsymbol{\gamma}\gamma_0\gamma_5 + \Delta_3\gamma_0\gamma_5, \tag{15}$$

where $\Delta_1, \Delta_2, \Delta_3$ are in general functions of $\boldsymbol{K}$, and

$$\boldsymbol{n} = \boldsymbol{q}/|\boldsymbol{q}|. \tag{16}$$

In this case when we expand to order Δ^3 (9) simplifies to

$$\Delta(\boldsymbol{n}', \boldsymbol{K}) = \frac{3d}{2\mu}\int\frac{\mathrm{d}\Omega}{4\pi}D_{AB}(\boldsymbol{n}, \boldsymbol{n}')\tilde{\Gamma}^A\gamma_0\gamma_5(q+m)\Gamma^B\left(a + bd^*d - \frac{c\beta^2}{16\mu^2}(\boldsymbol{q}\cdot\boldsymbol{K})^2\right)d \tag{17}$$

where

$$\boldsymbol{n}' = \boldsymbol{k}/|\boldsymbol{k}|, \tag{18}$$

$$D_{AB}(\boldsymbol{n}, \boldsymbol{n}') = D_{AB}(\boldsymbol{k}-\boldsymbol{q})|_{|\boldsymbol{k}|=|\boldsymbol{q}|=p_F}, \tag{19}$$

$$d \equiv \Delta_1 - p_F\Delta_2/\mu - m\Delta_3/\mu \tag{20}$$

and a, b and c are the constants

$$a = \frac{g^2}{6}\frac{\mathrm{d}n}{\mathrm{d}\varepsilon}\ln\zeta\beta\varepsilon_0, \qquad b = \frac{-7g^2\zeta(3)}{48(\pi k_B T_c)^2}\frac{\mathrm{d}n}{\mathrm{d}\varepsilon}, \qquad c = \frac{7g^2\zeta(3)}{6\pi^2}\frac{\mathrm{d}n}{\mathrm{d}\varepsilon}. \tag{21}$$

Much of the content of (17) can be extracted without specifying the exact form of the pairing interaction. We may isolate d from (17) by taking traces, and projecting the $J=0$ part by replacing K^iK^j by $\frac{1}{3}\delta^{ij}\boldsymbol{K}^2$. Then

$$d = [a + bd^*d - c(\beta^2 p_F^2/48\mu^2)\boldsymbol{K}^2]da_c^{-1} \tag{22}$$

with

$$a_c^{-1} = \frac{3}{8\mu^2}\int\frac{\mathrm{d}\Omega'}{4\pi}\int\frac{\mathrm{d}\Omega}{4\pi}D_{AB}(\boldsymbol{n}, n')$$
$$\times \mathrm{Tr}[(\mu\gamma_5 + p_F\boldsymbol{n}'\cdot\boldsymbol{\gamma}\gamma_0\gamma_5 + m\gamma_0\gamma_5)\tilde{\Gamma}^A\gamma_0\gamma_5(q+m)\Gamma^B]. \tag{23}$$

Equation (22) is the Ginzburg–Landau equation deriving from the Ginzburg–Landau free energy

$$\mathscr{F} = \frac{1}{2}\frac{\mathrm{d}n}{\mathrm{d}\varepsilon} t d^* d + \frac{7\zeta(3)\mu p_F}{32\pi^4 (k_B T_c)^2} (d^* d)^2 + \frac{7\zeta(3) p_F^3}{96\pi^4 \mu (k_B T_c)^2} \nabla d^* \cdot \nabla d, \tag{24}$$

where

$$t = (T - T_c)/T_c \tag{25}$$

and we have used

$$\mathrm{d}n/\mathrm{d}\varepsilon = \mu p_F/\pi^2. \tag{26}$$

(The overall normalisation may be checked by the evaluation of vacuum bubbles.)

If we specify the form of the pairing interaction we may obtain the detailed gap matrix. For example, we may assume scalar phonon exchange, $\Gamma^A = \tilde{\Gamma}^A = I$, and $D_{AB}(\boldsymbol{n}, \boldsymbol{n}') = D(\boldsymbol{n} \cdot \boldsymbol{n}')$. Then equation (17) leads to the gap matrix

$$\Delta(\boldsymbol{n}, \boldsymbol{K}) = \frac{\mu d(\mu\gamma_5 V_0 + p_F V_1 \boldsymbol{n} \cdot \boldsymbol{\gamma}\gamma_0\gamma_5 - m V_0 \gamma_0\gamma_5)}{(\mu^2 + m^2) V_0 - p_F^2 V_1} \tag{27}$$

with

$$V_l = -2p_F^2 \int \mathrm{d}\Omega (4\pi)^{-1} P_l(\boldsymbol{n} \cdot \boldsymbol{n}') D(\boldsymbol{n} \cdot \boldsymbol{n}') \tag{28}$$

and d determined by minimising (24).

Coupling to the electromagnetic field is introduced in the usual way by the substitution in equation (24)

$$\nabla \to \nabla \pm iq\boldsymbol{A} \tag{29}$$

according as the derivative acts on d or d^*, where $q = 2e$ is the charge of the Cooper pair. The terms

$$\mathscr{F}_{\mathrm{mag}} = \boldsymbol{B}^2/2\mu_0 - \boldsymbol{B} \cdot \boldsymbol{H}_0 + \tfrac{1}{2}\mu_0 \boldsymbol{H}_0^2 \tag{30}$$

must also be added in the presence of an applied magnetic field. Proceeding as in the non-relativistic case (see for example Tinkham 1975), we obtain

$$B_c^2 = 4\mu p_F (k_B T_c)^2 / 7\zeta(3) \tag{31}$$

and

$$\kappa^2 = \frac{1}{2}\frac{B_{c_2}^2}{B_c^2} = \frac{18\pi^3}{7\zeta(3)(e^2/4\pi)} \left(\frac{k_B T_c}{\mu}\right)^2 \left(\frac{\mu}{p_F}\right)^5 \tag{32}$$

(in units where $\hbar = c = \mu_0 = 1$, $e^2/4\pi = 1/137$). In the ultra-relativistic limit $\mu \approx p_F$ and we have $\kappa^2 \ll \frac{1}{2}$, i.e. type I superconductivity, if $(k_B T_c/\mu) \ll 10^{-2}$.

In the case of a good type I superconductor, we may neglect fluctuations in d compared with fluctuations in $\boldsymbol{A}$, and treat d as a spatially independent constant in the critical region. Integrating out the fluctuations in $\boldsymbol{A}$ just as in the non-relativistic case (Halperin *et al* 1974) leads to the effective free energy functional

$$\mathscr{F}_{\mathrm{eff}}(d) = \tilde{\alpha}|d|^2 - \xi|d|^3 + \bar{\beta}|d|^4 \tag{33}$$

with

$$\tilde{\alpha} = \tfrac{1}{2}(\mathrm{d}n/\mathrm{d}\varepsilon)(T - \tilde{T}_c)/\tilde{T}_c, \tag{34}$$

where $\tilde{T}_c$ differs from T_c because of fluctuations,

$$\bar{\beta} = \tfrac{7}{32}\zeta(3)\mu p_F/\pi^4(k_B T_c)^2 \tag{35}$$

and

$$\xi = \frac{\sqrt{2}}{3}\frac{k_B T_c}{\pi}\left(\frac{q^2 p_F^2 \bar{\beta}}{3\mu^2}\right)^{3/2}. \tag{36}$$

A first-order phase transition occurs at the temperature $T = \tilde{T}_c$ ($\neq \tilde{T}_c$) given by

$$\tilde{\alpha} = \xi^2/4\bar{\beta} \tag{37}$$

and at the phase transition

$$|d|_{T=\tilde{T}_c} = \xi/2\bar{\beta}. \tag{38}$$

Equation (33) is of the same form as for the discussion of the colour superconducting phase transition in quark matter (Bailin and Love 1981c) but with different expressions for the coefficients $\tilde{\alpha}$, $\bar{\beta}$ and ξ. Those results presented in our earlier paper which depend only on the parameter

$$\varepsilon = (-\beta\tilde{\alpha}/\xi^2)_{T=0} \tag{39}$$

may be carried over provided we use the modified vaiue

$$\varepsilon = \frac{243\pi^5}{784\zeta(3)^2(e^2/4\pi)^3}\left(\frac{k_B T_c}{\mu}\right)^2\left(\frac{\mu}{p_F}\right)^7. \tag{40}$$

In particular,

$$|d|_{T=\tilde{T}_c}/|d|_{T=0} = \tfrac{4}{3}[1+(1+\tfrac{32}{9}\varepsilon)^{1/2}]^{-1}. \tag{41}$$

In the non-relativistic limit, ε is very large and the phase transition is weakly first order. However, in the ultra-relativistic limit ($\mu \approx p_F$) we have $\varepsilon \lesssim 1$ when $(k_B T_c/\mu) \lesssim 10^{-4}$, and the transition may well be strongly first order.

Acknowledgments

This research was supported in part by the Science Research Council. We would like to thank Dr B R Gyorffy for a stimulating discussion.

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